

B-CONVEXITY AND REFLEXIVITY IN BANACH SPACES

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ABSTRACT. A proof of James that uniformly nonsquare spaces are reflexive is extended in part to *B*-convex spaces. A condition sufficient for non-*B*-convexity and related conditions equivalent to non-*B*-convexity are given. The following theorem is proved: A Banach space is *B*-convex if each subspace with basis is *B*-convex.

0. Introduction. The notion of a *B*-convex Banach space was introduced by A. Beck [1], [2] as a characterization of those Banach spaces *X* having the property that a certain strong law of large numbers holds for *X* valued random variables.

Definition. Let *k* be a positive integer and ϵ a positive number. *X* is said to be *k*, ϵ -convex if for any $\{x_1, \dots, x_k\}$, $\|x_i\| \leq 1$, $i = 1, \dots, k$, there is some choice of signs ξ_1, \dots, ξ_k so that $\|\sum_{i=1}^k \xi_i x_i\| \leq k(1 - \epsilon)$. *X* is said to be *B*-convex if it is *k*, ϵ -convex for some *k* and ϵ .

Further study of *B*-convex spaces has been done by R. C. James [6], [7], D. P. Giesy [5] and C. A. Kottman [8]. Giesy showed that *B*-convex spaces have many of the properties of reflexive spaces. James conjectured that all *B*-convex spaces are reflexive, and proved the conjecture true for 2, ϵ -convex spaces. Both James and Giesy proved the conjecture true for *B*-convex spaces having an unconditional basis. Kottman extended James' 2, ϵ -convex proof to a larger subclass, *P*-convex spaces. Examples are known of spaces which are reflexive but not *B*-convex.

§1 of this paper adopts a part of James' 2, ϵ -convex theorem to all non-*B*-convex spaces, presents a condition sufficient for non-*B*-convexity, and gives related characterizations of non-*B*-convex spaces, though the conjecture of James remains open. §2 proves a theorem on *B*-convexity and subspaces with basis analogous to a theorem of Pełczyński on reflexivity and subspaces with basis.

For a Banach space *X*, *U*(*X*) will denote the closed unit ball $\{x: \|x\| \leq 1\}$ of *X*.

I. Non-*B*-convexity. In James' proof [6] that 2, ϵ -convex spaces are reflexive, he defines for a Banach space *X* a sequence of numbers K_n , and shows that if *X*

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is not reflexive then $K_n \leq 2n$, and in that case X cannot be $2, \epsilon$ -convex. We will extend the second step of this result to show that if K_n is a bounded sequence then X cannot be B -convex. The numbers K'_n , to be defined almost the same as James' K_n , will be used instead of K_n . Another condition, which implies that $\{K'_n\}$ is bounded, is introduced and is shown to be sufficient for non- B -convexity by a much simpler proof.

Let X be a Banach space. For each sequence $\{f_j\}$ of continuous linear functionals with unit norms and each increasing sequence of integers $\{p_1, \dots, p_{2n}\}$, let $S(p_1, \dots, p_{2n}, \{f_j\})$ denote the set of all x such that, for all k and i , $3/4 \leq (-1)^{i-1} f_k(x)$ if $p_{2i-1} \leq k \leq p_{2i}$ and $1 \leq i \leq n$. Let

$$K(n, \{f_j\}) = \liminf_{p_1 \rightarrow \infty} \liminf_{p_2 \rightarrow \infty} \dots \liminf_{p_{2n} \rightarrow \infty} \inf \{\|z\| : z \in S(p_1, \dots, p_{2n}, \{f_j\})\}$$

and

$$K'_n = \inf \{K(n, \{f_j\}) : \|f_j\| = 1 \text{ for all } j\}.$$

James' definition of K_n is similar. It follows from the definitions that $K'_n \leq K_n$ and $K'_n \leq K'_{n+1}$ for all n .

Theorem 1.1. *If the sequence $\{K'_n\}$ for a Banach space X is bounded, then X is not B -convex.*

Proof. For any positive integer k and any $0 < \delta < 2$ we will show X is not k, δ -convex by showing there are $x_1, \dots, x_k \in U(X)$ such that for any choice of signs ξ_1, \dots, ξ_k we have $\|\sum_{i=1}^k \xi_i x_i\| > k(1 - \delta)$. Since the sequence $\{K'_n\}$ is bounded, and monotone, we can choose m such that $K'_{2m}/K'_{3m2k} > 1 - \delta/3$. Let $3m2^k = M$. Choose $\mu, \{f_j\}$ where $\|f_j\| = 1$, and ϵ such that $0 < \mu < (K'_M)^2 \delta/3K'_{2m}$, $K'_M + \mu > K(M, \{f_j\})$ and $0 < \epsilon < (K'_M)^2 \delta/3(K'_{2m} + K'_M)$. From these inequalities, it follows that

$$(K(2m, \{f_j\}) - \epsilon)/(K(M, \{f_j\}) + \epsilon) > 1 - \delta.$$

As will be described below, it is possible to choose an increasing set of integers $P = \{p_{i,j} : i = 1, \dots, k; j = 1, \dots, 2M\}$ having the following properties:

- (1) For each $i = 1, \dots, k$ there is $u_i \in S(p_{i,1}, \dots, p_{i,2M}; \{f_j\})$ such that $\|u_i\| \leq K(M, \{f_j\}) + \epsilon$.
- (2) For each choice of signs ξ_1, \dots, ξ_k there is an increasing set of integers $\{\sigma_1, \dots, \sigma_{4m}\} \subset P$ such that
 - (2a) $(1/k) \sum_{i=1}^k \xi_i u_i \in S(\sigma_1, \dots, \sigma_{4m}; \{f_j\})$, and
 - (2b) any element of $S(\sigma_1, \dots, \sigma_{4m}; \{f_j\})$ has norm greater than or equal to $K(2m, \{f_j\}) - \epsilon$.

Let $x_i = u_i / K(M, \{f_j\}) + \epsilon$ for $i = 1, \dots, k$. From property (1), $\|x_i\| \leq 1$. From property (2), for any choice of signs ξ_1, \dots, ξ_k we have

$$\left\| \frac{1}{k} \sum_{i=1}^k \xi_i x_i \right\| \geq \frac{K(2m, \{f_j\}) - \epsilon}{K(M, \{f_j\}) + \epsilon} > 1 - \delta$$

which completes the proof except for the choice of P .

The choice of P is rather tedious. Integers are chosen successively in m blocks of increasing integers:

$$\begin{aligned} P_1 &= \{p_{i,j} : i = j, \dots, k; j = 1, \dots, 2M/m\} \\ P_2 &= \{p_{i,j} : i = 1, \dots, k; j = (2M/m) + 1, \dots, 4M/m\} \\ &\vdots \\ P_m & \end{aligned}$$

Let the k -tuples of signs (ξ_1, \dots, ξ_k) be denoted Ξ_1, \dots, Ξ_{2^k} . The integers of P_1 are chosen successively in 2^k sets of increasing integers $P_1(\Xi_1), \dots, P_1(\Xi_{2^k})$. The number of integers in $P_1(\Xi_n)$ depends on Ξ_n ; as will be shown, four are chosen for each plus sign in Ξ_n and eight for each minus sign, so that P_1 has $6k2^k = 2kM/m$ integers.

Property (2a) is provided by the order of choice of the integers. This order may be illustrated by supposing $\Xi_n = (\xi_1, \dots, \xi_k)$ where $\xi_\eta = -1$ ($\eta \neq 1$ or k) and $\xi_i = +1$ for $i \neq \eta$. Suppose for each $i = 1, \dots, k$, the last integers of $P_1(\Xi_{n-1})$ are $p_{i,j(i)}$. The order of choice of integers of $P_1(\Xi_n)$ is shown in Figure 1. The integers $\sigma_1, \dots, \sigma_4$ of property (2) for this choice of signs are $p_{k,j(k)+1}, p_{1,j(1)+1}, p_{k,j(k)+3}, p_{1,j(1)+4}$. The sets P_2, P_3, \dots, P_m are chosen in succession by the same method. The integers $\sigma_1, \dots, \sigma_{4m}$ for a choice of signs Ξ_n are in the set $\{P_1(\Xi_n), \dots, P_m(\Xi_n)\}$. Properties (1) and (2b) are provided by requiring that the integers chosen satisfy appropriate inequalities at each step.

The proof of Theorem 1.1 is rather tedious. We now present a strengthening of the condition " $\{K'_n\}$ bounded" which will be sufficient for non-B-convexity in a much simpler way.

Condition 1. For some $0 < \epsilon < 1$, $U(X^*)$ contains a sequence $\{f_j\}$ such that for any $m \geq 1$ and any choice of signs ξ_1, \dots, ξ_m there is $x \in U(X)$ satisfying $f_j(\xi_j x) > \epsilon$, $j = 1, \dots, m$.

Theorem 1.2. If a Banach space satisfies Condition 1, then the sequence $\{K'_n\}$ for that space is bounded.

Proof. We will show $K'_n \leq 3/4\epsilon$ by showing that for $\{f_j\}$ as asserted in Con-

Figure 1. Order of choice of $P_1(\Xi_n)$

For order of choice, read down first column, then down second column, etc.

$$\begin{array}{cccc}
 P_{1, j(1)+1} & P_{1, j(1)+2} & P_{1, j(1)+3} & P_{1, j(1)+4} \\
 \vdots & \vdots & \vdots & \vdots \\
 P_{\eta-1, j(\eta-1)+1} & P_{\eta-1, j(\eta-1)+2} & P_{\eta-1, j(\eta-1)+3} & P_{\eta-1, j(\eta-1)+4} \\
 P_{\eta, j(\eta)+1} & & & \\
 P_{\eta, j(\eta)+2} & & & \\
 P_{\eta, j(\eta)+3} & P_{\eta, j(\eta)+4} & P_{\eta, j(\eta)+5} & P_{\eta, j(\eta)+6} \\
 & & & P_{\eta, j(\eta)+7} \\
 & & & P_{\eta, j(\eta)+8} \\
 P_{\eta+1, j(\eta+1)+1} & P_{\eta+1, j(\eta+1)+2} & P_{\eta+1, j(\eta+1)+3} & P_{\eta+1, j(\eta+1)+4} \\
 \vdots & \vdots & \vdots & \vdots \\
 P_{k, j(k)+1} & P_{k, j(k)+2} & P_{k, j(k)+3} & P_{k, j(k)+4}
 \end{array}$$

dition 1, and any p_1, \dots, p_{2n} there is $y \in S(p_1, \dots, p_{2n}; \{f_j / \|f_j\|\})$ and $\|y\| \leq 3/4\epsilon$. Choose ξ_k , $k = 1, \dots, p_{2n}$, so that if $p_{2i-1} \leq k \leq p_{2i}$, $1 \leq i \leq n$, then $\xi_k = (-1)^{i-1}$. Let $y = 3x/4\epsilon$, where x is that element asserted by Condition 1.

Theorem 1.3. *The following conditions are equivalent to non-B-convexity:*

Condition 2. *For some $0 < \epsilon < 1$ and any $k \geq 2$, $U(X^*)$ contains f_1, \dots, f_k such that for any choice of signs ξ_1, \dots, ξ_k there is $x \in U(X)$ satisfying $f_j(\xi_j x) > \epsilon$, $j = 1, \dots, k$.*

Condition 3. *For some $0 < \epsilon < 1$ and any $k \geq 2$, $U(X)$ contains x_1, \dots, x_k such that for any choice of signs ξ_1, \dots, ξ_k there is $f \in U(X^*)$ satisfying $f(\xi_j x_j) > \epsilon$, $j = 1, \dots, k$.*

Proof. We first show non-B-convexity implies Condition 2. Choose any $0 < \epsilon < 1$. For a given $k \geq 2$ choose δ such that $k - 1 + \epsilon < k(1 - 2\delta)$. Since X is not B-convex, by [5, Theorem II-3], X^* is not B-convex. Hence there is $f_1, \dots, f_k \in U(X^*)$ such that for all choices of signs ξ_1, \dots, ξ_k , $\|\sum_{i=1}^k \xi_i f_i\| > k(1 - \delta)$. Thus, for each choice of signs there is $x \in U(X)$ such that $\sum_{i=1}^k \xi_i f_i(x) > k(1 - 2\delta)$. If there were some j such that $f_j(\xi_j x) \leq \epsilon$, then $\sum_{i=1}^k \xi_i f_i(x) \leq k - 1 + \epsilon < k(1 - 2\delta)$.

To prove the converse, suppose X is B-convex and that Condition 2 is satisfied. Since X^* is also B-convex, by [5, Lemma I-4], $\lim_{k \rightarrow \infty} a_k(X^*) = 0$, where

$$a_k(X^*) = \sup \left\{ \inf \left\{ \frac{1}{k} \sum_{i=1}^k \xi_i x_i : \xi_1, \dots, \xi_k = \pm 1 \right\} : f_1, \dots, f_k \in U(X^*) \right\}$$

and we may choose k so that $a_k(X^*) < \epsilon$. Therefore, for f_1, \dots, f_k asserted by Condition 2 there is some choice of signs such that $\|\sum_{j=1}^k \xi_j f_j\| < k\epsilon$. But by Condition 2, $f_j(\xi_j x) > \epsilon$, so that $\sum_{j=1}^k \xi_j f_j(x) > k\epsilon$ and $\|\sum_{j=1}^k \xi_j f_j\| > k\epsilon$ which is a contradiction.

Condition 3 may be shown equivalent to non- B -convexity by similar proofs.

Corollary 1.4. *If a Banach space satisfies Condition 1 it is not B -convex.*

We do not know whether nonreflexivity implies $\{K'_n\}$ bounded or Condition 1. There is however a large class of nonreflexive, non- B -convex spaces satisfying Condition 1, listed in the following easily proved proposition.

Proposition 1.5. *The Banach spaces c_0 , l_1 , and all spaces containing c_0 or l_1 satisfy Condition 1.*

2. B -convexity and basis. It is known that a Banach space is reflexive if each subspace with basis is reflexive [9]. In this section we show that a Banach space is B -convex if each subspace with basis is B -convex.

As usual we say that $\{x_i\} \subset X$ is a basis for X if for each $x \in X$ there is a unique sequence of numbers $\{a_i\}$ so that $\lim_{n \rightarrow \infty} \|\sum_{i=1}^n a_i x_i - x\| = 0$. It is well known that $\{x_i\}$ is a basis for X if there is some number k so that for any integers n and q and any sequence of numbers $\{a_i\}$ we have

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq k \left\| \sum_{i=1}^{n+q} a_i x_i \right\|.$$

We will prove the main theorem of this section by constructing a basic sequence in an arbitrary non- B -convex space in such a way that the span of this sequence is not B -convex. The technique for construction of this basic sequence is an adaptation of the method of Day [3] and Gelbaum [4]. It relies on the following lemmas:

Lemma 2.1. *If X is finite dimensional, for any $\epsilon > 0$ there is $\{f_i\}_{i=1}^n \subset U(X^*)$ such that, for any x ,*

$$\|x\| \leq (1 + \epsilon) \max \{f_i(x) : i = 1, \dots, n\}.$$

Lemma 2.2. *If $\{f_i\}_{i=1}^n \subset X^*$ and $Y = \bigcap_{i=1}^n f_i^{-1}(0)$, then Y is a space of finite codimension in X ; that is, there is a finite dimensional subspace Z so that $X = Y \oplus Z$.*

Theorem 2.3. *If X is not B -convex it contains a subspace with basis not B -convex.*

Proof. Let $\{\delta_i\}$ be a sequence of positive numbers less than one tending to zero. Let $\{k_i\}$ be a sequence of integers tending to infinity. Let $p(0) = 0$, $p(m) = \sum_{i=1}^m k_i$, $m = 1, 2, \dots$.

The subspace to be constructed will be the closed span of a sequence $\{x_i\}$ having the following properties:

(1) there is a sequence $\{\epsilon_i\}$ tending to zero so that for each $m = 1, 2, \dots$, the space $[x_i]_{i=p(m-1)+1}^{p(m)}$ is an ϵ_m isometric image of $l_1^{k_m}$, in particular,

$$(1 - \epsilon_m) \sum_{i=p(m-1)+1}^{p(m)} |a_i| \leq \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i|.$$

(2) For any $\{a_i\}$, n, q ,

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq (3 + \delta_n) \left\| \sum_{i=1}^{n+q} a_i x_i \right\|.$$

Property (1) shows that the subspace is not B -convex, and property (2) shows that $\{x_i\}$ is a basis.

For $\{\delta_i\}$, $\{k_i\}$ as above there are $\{\epsilon_i\}$, $\{\eta_i\}$ such that

(3) $\epsilon_i \rightarrow 0$.

(4) If $1 \leq n \leq p(1)$ then $(1 + \eta_1)/(1 - \epsilon_1) \leq 1 + \delta_n$.

(5) If $p(m) < n \leq p(m+1)$ then

$$1 + \eta_m + (2 + \eta_m + \eta_{m+1})/(1 - \epsilon_{m+1}) \leq 3 + \delta_n.$$

$[x_i]_{i=1}^{p(m)}$ will be denoted by $L_{p(m)}$. $\{x_i\}$ will be constructed in blocks by induction on m . In the induction step, from a previously constructed subspace Λ_{m-1} , $\{x_i\}_{i=p(m-1)+1}^{p(m)}$ will be chosen satisfying (1). Then a subspace Λ_m of Λ_{m-1} , of finite codimension in X , will be constructed so that $\Lambda_m \cap L_{p(m)} = 0$ and the projection $P_m: \Lambda_m \oplus L_{p(m)} \rightarrow L_{p(m)}$ satisfies $\|P_m\| < 1 + \eta_m$. Finally (2) will be proved.

Let $m = 1$. Since X is not B -convex, we can choose $\{x_i\}_{i=1}^{k_1=p(1)} \subset U(X)$ satisfying (1). To construct Λ_1 choose $\{f_i\}_{i=1}^{q(1)} \subset U(L_{p(1)}^*)$ by Lemma 2.1 and extend them to X without increase of norm so that if $x \in L_{p(1)}$,

$$\|x\| \leq (1 + \eta_1) \max\{f_i(x): i = 1, \dots, q(1)\}.$$

Let $\Lambda_1 = \bigcap_{i=1}^{q(1)} f_i^{-1}(0)$. By Lemma 2.2, Λ_1 is of finite codimension in X , and $L_{p(1)} \cap \Lambda_1 = 0$, so there is a projection $P_1: L_{p(1)} \oplus \Lambda_1 \rightarrow L_{p(1)}$. To see that $\|P_1\| \leq 1 + \eta_1$ we have for any $x \in L_{p(1)}$, $\lambda \in \Lambda_1$, and some $i = 1, \dots, q(1)$, $\|P(x + \lambda)\| = \|x\| \leq (1 + \eta_1)f_i(x) = (1 + \eta_1)f_i(x + \lambda) \leq (1 + \eta_1)\|x + \lambda\|$.

Now suppose we have $\{x_i\}_{i=1}^{p(1)}$ satisfying (1), $\{f_i\}_{i=1}^{q(m-1)} \subset U(X^*)$, and for

$n = 1, 2, \dots, m-1$, $\Lambda_n = \bigcap_{i=1}^{q(n)} f_i^{-1}(0)$, such that $\|P_n\| \leq 1 + \eta_n$ where $P_n: \Lambda_n \oplus L_{p(n)} \rightarrow L_{p(n)}$. Since Λ_{m-1} is of finite codimension in X , by [5, Theorem 2.12], it is not B -convex so that there are $\{x_i\}_{i=p(m-1)+1}^{p(m)} \subset U(\Lambda_{m-1})$ satisfying (1). By Lemma 2.1 choose $\{f_i\}_{i=q(m-1)+1}^{q(m)} \subset U(X^*)$ so that if $x \in L_{p(m)}$,

$$\|x\| \leq (1 + \eta_m) \max \{f_i(x): i = q(m-1) + 1, \dots, q(m)\}.$$

Let $\Lambda_m = \bigcap_{i=1}^{q(m)} f_i^{-1}(0)$. Then exactly as in the $m = 1$ case, $L_{p(m)} \cap \Lambda_m = 0$ and $\|P_m\| \leq 1 + \eta_m$.

To show (2) holds we first observe, for any $a_i, i = 1, 2, \dots$,

A. $\|\sum_{i=1}^{p(m)} a_i x_i\| \leq (1 + \eta_m) \|\sum_{i=1}^{p(m)+q} a_i x_i\|$ for any $q = 1, 2, \dots$, since $\|P_m\| \leq 1 + \eta_m$.

Further, we observe

B. If $p(m-1) < n \leq p(m)$ then

$$\left\| \sum_{i=p(m-1)+1}^n a_i x_i \right\| \leq \frac{1}{1 - \epsilon_m} \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\|,$$

since

$$\left\| \sum_{i=p(m-1)+1}^n a_i x_i \right\| \leq \sum_{i=p(m-1)+1}^n |a_i| \leq \sum_{i=p(m-1)+1}^{p(m)} |a_i| \leq \frac{1}{1 - \epsilon_m} \left\| \sum_{i=p(m-1)+1}^{p(m)} a_i x_i \right\|,$$

using the inequalities of (1).

(2) can be proved in four cases as follows:

Case 1. $1 \leq n < n + q < p(1)$. Using B, with $m = 1$, and (4) we obtain

$$\|\sum_{i=1}^n a_i x_i\| \leq (1 + \delta_n) \|\sum_{i=1}^{n+q} a_i x_i\|.$$

Case 2. $1 \leq n \leq p(1) < n + q$. The above inequality can be proved using B and A, where $m = 1$, and (4).

Case 3. There is m so that $p(m) < n < n + q \leq p(m+1)$. Inequality (2) can be proved by using A, B with m replaced by $m+1$, and (5).

Case 4. There is m so that $p(m) < n \leq p(m+1) < n + q$. Inequality (2) can be proved by using A, B with m replaced by $m+1$, A with m replaced with $m+1$, and (5).

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